Optimal-Maintenance Modeling on Finite Time with Technology Replacement and Changing Repair Costs

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SUMMARY & CONCLUSIONS

We explore maintenance models for finite time missions. Motivated by real world applications, and models which are lacking in literature, we include exponential cost forms that allow for net present value analysis, and explore replacing obsolete equipment with new.

In this paper, we demonstrate through mathematical modeling that:

- By using simple search methods, we can find optimal replacement intervals for systems under a general finite time mission.
- Under Weibull hazard functions, we can express the optimal replacement rate simply.
- By allowing for special exponential cost function forms, we can find local cost minimums for exponentially growing costs, or include the time value of money.
- For systems in use, we can monitor the marginal cost, and replace the system when this marginal cost equals the expected average cost for the replacing system.
- When replacing one technology with a new, there are conditions under which the optimal replacement cycles are completely interrelated, and other conditions where we can calculate one from an independent calculation of another.
- In budget constrained times, use present value estimates such as the ones we provide.

To apply optimal replacement planning in industry, we must consider finite time missions, net present value of costs, and replacing obsolete equipment with new technology.

1. INTRODUCTION

Optimal maintenance intervals often neglect three important facts of real world applications:

- The equipment may be needed for a finite time mission regardless of the replacement intervals (finite contract maintenance, or maintaining subsystems in a rented system),
- The value of money changes over time, and can affect optimal replacement intervals, and
- Often new technology allows the replacement to be more reliable and/or cheaper.

Each of these facts affects the planned replacement interval. While some authors consider finite time missions, their mathematical approaches do not include technology replacement or cost accounting. I will present some important results found by exploring optimal maintenance models under these new paradigms.

Companies with large amounts of equipment to maintain can often fall behind in maintenance. When optimal maintenance policies dictate more replacements than the current cash flow can support, the problem changes to selecting the best equipment to replace. We touch on the selection of equipment for replacement based on predicted maintenance cost.

Notation

$C(t)$	Total expected cost of a system over a time interval t.
T	Cycle time for replacement.
t	Time.
$E[N_i(t)]$	Expected number of minimal repairs $(i=1)$, or replacements $(i=2)$.
C_i	Cost of a minimal repair $(i=1)$, or replacement $(i=2)$.
$C_i(t)$	Cost of a minimal repair $(i=1)$, or replacement $(i=2)$, as a function of time.
$h(t)$	Hazard function, hazard rate, or failure rate as a function of time.
$H(t)$	Cumulative hazard function, hazard rate, or failure rate as a function of time.
N	Number of cycles.
T_h	Finite time horizon, end time for the interval $[0,T_h]$.
$T_r,\overline{T}_r$	Replacement time interval for an old and new technology, respectively.
$TC(T_r)$	Total expected cost over a planning horizon T_h, as a function of T_r.
i	Discount rate, annual cost of money. May also be a parameter for a general cost function.
m	Number of planned replacement cycles for a new technology.
$C(T_r,\overline{T}_r)$	Total expected cost per unit of time for a cycle of old technology, and m of a new.
$TC(T_r,\overline{T}_r)$	Total expected cost over a planning horizon, a function of T_r, and $\overline{T}_r$.

EC_{ild}, EC_{new}	Expected cost over a planning horizon, of maintaining old equipment, or replacing with new, respectively.

Nomenclature

DFR	Decreasing Failure Rate
IFR	Increasing Failure Rate
NPV	Net Present Value
Minimal repair	Return of a system to a functional state without affecting the hazard rate.
Replacement	A complete reconditioning of the system so that the system is as good as new.

Assumptions

1. The system is subject to failures that technicians minimally repair as needed.
2. Technicians can replace the existing system with a new system.
3. The times for minimal repair and replacement are negligible.
4. The cost of a replacement is constant, when ignoring the discount rate.
5. The repair cost rate times the hazard rate is increasing in time.
6. A new system begins in a working state, with no failures at age zero.
7. The repair cost and hazard rates are continuous functions in time.

In addition, depending on the model, we may apply a subset of these *optional assumptions*:

1. The hazard rate is Weibull distributed.
2. The cost of a minimal repair is constant.
3. The cost of a minimal repair is a function of time.
4. The cost of a minimal repair is a particular exponential form function of time.
5. The cost of a minimal repair is discounted for NPV.
6. Equipment will be in service for a predetermined amount of time.
7. Equipment will be replaced by a new technology, which will serve for a known number of replacement cycles.
8. Equipment will be replaced by a new technology, which will serve until a predetermined future time.
9. There are always sufficient resources to replace equipment at the optimal time.
10. There are not sufficient resources to replace all equipment at the optimal time.

2. BACKGROUND

A wealth of literature exists on the topic of optimal replacement strategies. Two very complete surveys of the relevant developments in this field are worth mentioning here: Pierskalla & Voelker (Ref. 9), and Valdez-Florez & Feldman (Ref. 11). Many models in the literature make common assumptions:

- A minimal repair does not change the condition of the system.
- A replacement is a complete recondition of the system so that the system is as good as new.
- For the system, the modeler understands well the repair cost function and the replacement cost function. Often, cost is a function of time or events that occur over time, or it may also be constant.
- When cost is constant, modelers often assume the failure rate is increasing.

There is significant literature covering infinite time horizon models. As early as 1960, Barlow and Hunter (Ref. 2) investigated the ideas of periodic replacement with failures addressed through minimal repair between the replacement events. They developed a cost function over an infinite horizon for a system with IFR, and a periodic renewal. More recently, Deshpande and Singh (Ref. 7) developed a general model for a minimally repaired system with a changing cost rate that is both a function of the repair time, and the number of repairs on the system. Bagai and Jain (Ref. 1) constructed a similar set of models. Although similar to Refs. 2 and 7, they allowed for probabilistic replacement of a failed system. Nakagawa and Kijima (Ref. 8) presented another model for a system with minimal repairs and replacements. But this model assumes replacements occur at optimal times, shock events, or damage events. Puri and Singh (Ref. 10) developed models with general cost functions, and provided a useful lemma.

Very few models for finite horizons exist. Boland and Proschan (Ref. 5) created a model for optimal replacement with linear minimal repair cost, all over a finite time period, and then extended this for an infinite time period. They restrict the cost function to be a linear function of the number of minimal repairs. Boland and Proschan (Ref. 6) revisited this work in terms of system shocks, but concentrate more on infinite time horizons. Boland (Ref. 4) conducted similar work alone.

While finite horizon models are not new, they are underdeveloped in the literature. Also, I could not find any literature specifically addressing exponential cost functions, or models taking into account the time value of money. Likewise, I could not find any development on the concept of replacing an old technology with a new one having different costs and hazard behavior. Many industries experience each of these constraints, individually and in combination.

3. AGE REPLACEMENT UNDER A FINITE TECHNOLOGY HORIZON

3.1. Constant Cost

The total expected cost over a time horizon for installing and maintaining a system subject to failure and periodic replacement is

$$TC(T_r) = \frac{T_h}{T_r}\left[\int_0^{T_r} C_1(u)h(u)du + C_2\right] \qquad (1)$$

with T_r restricted so that T_h/T_r is any positive integer.

We can sometimes obtain the optimal replacement interval T_r by finding the absolute minimum for the continuous function eq (1) with T_r continuous, finding the two discrete values of T_r between which the absolute minimum resides, and choosing the minimum of these two values as the solution. The proof of this relies on two theorems which we can use to find optimal solutions in many of the models presented in this paper.

Theorem 1: The optimal replacement interval T_r is in the set of solutions such that T_h/T_r is any positive integer, under a finite time horizon and assuming the cost rate times the hazard rate is increasing in time.

Ref. 4 provides a proof of this theorem.

Theorem 2: The optimal replacement interval T_r is one of the two integer values x and z such that $x \le y \le z$, with y being the optimal replacement time found by allowing for continuous solutions with fractional setup costs. This applies when we consider a finite time horizon and the continuous cost function is convex.

The Appendix holds our proof of this.

When the two theorems are true, this approach will find the global optimal replacement time. In cases where the second theorem does not apply, say due to non-monotonic cost rate times hazard rate, we can still search all possible values for T_r by starting with T_h and decreasing until the time intervals become sufficiently small.

Assume that the minimal repair cost in eq (1) is constant. We now find the absolute minimum of eq (1) over the continuous set of values for T_r. Using Leibniz's rule,

$$\frac{\delta TC}{\delta T_r} = T_h\left(\frac{C_1 h(T_r)}{T_r} - \frac{C_1 H(T_r) + C_2}{T_r^2}\right) \tag{2}$$

and

$$\frac{\delta^2 TC}{\delta T_r^2} = T_h C_1\left(\frac{h'(T_r)}{T_r} - 2\frac{h(T_r)}{T_r^2} + 2\frac{H(T_r)}{T_r^3}\right) + 2\frac{T_h C_2}{T_r^3} \tag{3}$$

By rearranging the second order condition eq (3), the zero for the first order condition eq (2) provides a minimum solution when

$$\begin{aligned} &2(H(T_r) + C_2)T_r^{-4} > -h'(T_r)T_r^{-2} + 2h(T_r)T_r^{-3} \\ &= (-1)\frac{\delta}{\delta T_r}(h(T_r)T_r^{-3}) \end{aligned} \tag{4}$$

The left-hand side is positive. The right hand side will be negative if $h(T_t)$ is of higher order than T_r^3.

If the failure distribution is Weibull, then the first and second order conditions are, respectively,

$$C_1(\lambda^\beta(\beta-1)T_r^{\beta-2}) - \frac{C_2}{T_r^2} = 0 \tag{5}$$

$$\frac{T_h C_1}{T_r^3}\left((\lambda T_r)^\beta(\beta-2)(\beta-1)\right) + 2\frac{T_h C_2}{T_r^3} > 0 \tag{6}$$

The second order condition is positive for $\beta \ge 2$. For $\beta = 2$, the continuous minimum is

$$T_r = \sqrt{\frac{C_2}{C_1(\lambda^2)}} \tag{7}$$

3.2. Discounted or Exponential Repair Cost

In this case, we allow the repair cost to take on an exponential form. Not only can we include a discount rate in our model with this approach, but we can also consider other exponential cost function forms. The total expected cost over a time horizon in this case is

$$TC(T_r) = \frac{T_h}{T_r}\left[\int_0^{T_r} C_1(u)h(u)du + C_2\right] \tag{8}$$

and taking the same approach as in the model with constant repair cost, the first order condition is

$$h(T_r)C_1(T_r) - \frac{TC(T_r)}{T_h} = 0 \tag{9}$$

with $h(0)C_1(0) = 0$. Incorporating the first condition into the second yields the second order condition

$$\frac{T_h}{T_r}\left(h'(T_r)C_1(T_r) + h(T_r)C_1'(T_r)\right) > 0 \tag{10}$$

By our assumptions, $h(t) > 0$, $h'(t) > 0$, and $C_1(t) > 0$ for all *t*. If also $C_1'(t) > 0$, then we have at worst a local minimum. Otherwise, $C_1'(t)$ must meet the condition, over the entire range of T_r,

$$C_1'(T_r) > \frac{-C_1(T_r)h'(T_r)}{h(T_r)} \tag{11}$$

Because the general second order conditions do not guarantee a global minimum for all possible parameter values, we do not provide them here.

If we interpret the hazard rate as the distribution of cost realizations, then the first order equality says the optimal point exists where the total cost per unit of time is equal to the overall cost. This is equivalent, in cost model terms, to saying the average cost is equal to the marginal cost (AC=MC). Later in this section, we will discuss a replacement policy that relates to this concept.

Now lets investigate an exponential cost form with a Weibull hazard rate. Let $C_1(t) = C_1(1+i)^{at}$ where C_1 is constant, *i* is a rate of cost increase or decrease ($i \in (0,1)$), and *a* is 1 or –1 depending on whether the cost is increasing or decreasing with time. Also, let $h(t) = \lambda^\beta \beta t^{\beta-1}$ with β a positive integer. The total expected cost over the finite horizon is

$$TC(T_r) = \frac{T_h}{T_r}\left(\begin{array}{l} C_2 + C_1 \dfrac{\lambda^\beta \beta!}{(-a\ln(1+i))^\beta} \times \\ \left[1-(1+i)^{aT_r} \displaystyle\sum_{r=0}^{\beta-1} \frac{(-aT_r \ln(1+i))^r}{r!} \right] \end{array} \right) \qquad (12)$$

For non-integer β values, the total expected cost is not a closed form. We must use an approximation method, such as numerical integration.

The first order condition is quite messy, and not revealing. But the second order condition, after incorporating the first order condition, is simple

$$\frac{\beta-1}{T_r} > -a\ln(1+i) \qquad (13)$$

If the cost is increasing with time ($a=1$), and the hazard rate is increasing with time ($\beta \geq 1$), then we will find at worst a local minimum with the approach outlined in this section. If instead $a=-1$, then a local minimum exists in the range of T_r bounded by

$$T_r < \frac{\beta-1}{\ln(1+i)} \qquad (14)$$

Fortunately, to find the global optimal, we do not need the first order condition. Because T_r takes on discrete, defined values, we can find a local optimal solution by searching the possible values of T_r beginning with T_h and decreasing (Theorem 1). If we find a solution bounded by two higher cost solutions, we check the second order condition. If the second order condition shows this to be a global minimum, we are done. If the second order conditions are not met over the entire range of values for T_r, then we check the boundary values for optimal solutions. A plot of the reasonable values for T_r may also reveal the optimal solution over the discrete set of T_r. Small, non-zero value of T_r may not be implementable.

Now we leverage the development of eq (12) to change the cost function into an NPV adjustment. The total expected cost is then

$$\begin{array}{l} TC(T_r) = \dfrac{(1+i)^{T_r}}{(1+i)^{T_r}-1}\left[1-\dfrac{1}{(1+i)^{T_h}}\right] \times \\ \left[C_2 + \dfrac{C_1\lambda^\beta \beta!}{(\ln(1+i))^\beta}\left[1-(1+i)^{-T_r}\displaystyle\sum_{r=0}^{\beta-1}\frac{(T_r\ln(1+i))^r}{r!}\right]\right] \end{array} \qquad (15)$$

Notice that the cost is now a function of time, not the time since a new cycle. System renewal is not cost renewal in this case.

Now that the total expected costs for two cycles of equal length beginning at different times are not equal, conceivably the optimal cycle length could be a function of when the cycle starts, so that the optimal cycle length for any two different cycles may not be equal. However, the following theorem shows that the optimal cycle length is independent of when the cycle begins, so we only need calculate the one optimal cycle length.

Theorem 3: The optimal cycle time is equal for all cycles, regardless of the discount rate, and the time a given cycle begins.

Therefore, the optimal cycle time we find by developing eq (15) will apply to all cycles. We prove this theorem in the Appendix.

The first and second order conditions are, respectively,

$$\begin{array}{l} \left[1-(1+i)^{-T_h}\right] \times \\ \left[\begin{array}{l} \dfrac{(1+i)^{T_r}\ln(1+i)}{(1+i)^{T_r}-1}\left[1-\dfrac{(1+i)^{T_r}}{(1+i)^{T_r}-1}\right] \times \\ \left[\begin{array}{l} C_2 + \dfrac{C_1\lambda^\beta\beta!}{(\ln(1+i))^\beta} \times \\ \left(1-(1+i)^{-T_r}\displaystyle\sum_{r=0}^{\beta-1}\frac{(T_r\ln(1+i))^r}{r!}\right) \end{array} \right] \\ + \left[\dfrac{(1+i)^{T_r}}{(1+i)^{T_r}-1}\right]\left[\dfrac{C_1\lambda^\beta \beta T_r^{\beta-1}}{(1+i)^{T_r}}\right] \end{array} \right] = 0 \end{array} \qquad (16)$$

and

$$\begin{array}{l} \left[1-(1+i)^{-T_h}\right] \times \\ \left[\begin{array}{l} (\ln(1+i))^2 \left(\begin{array}{l} \dfrac{(1+i)^{T_r}}{(1+i)^{T_r-1}} - 3\left(\dfrac{(1+i)^{T_r}}{(1+i)^{T_r}-1}\right)^2 \\ + 2\left(\dfrac{(1+i)^{T_r}}{(1+i)^{T_r}-1}\right)^3 \end{array} \right) \\ \times \left(C_2 + \dfrac{C_1\lambda^\beta\beta!}{(\ln(1+i))^\beta} \left(\begin{array}{l} 1-(1+i)^{-T_r} \times \\ \displaystyle\sum_{r=0}^{\beta-1}\frac{(T_r\ln(1+i))^r}{r!} \end{array} \right) \right) \\ + \dfrac{C_1\lambda^\beta\beta T_r^{\beta-2}}{(1+i)^{T_r}-1} \left(\begin{array}{l} (\beta-1) - T_r\ln(1+i) \times \\ \left(1-2\dfrac{(1+i)^{T_r}}{(1+i)^{T_r}-1}\right) \end{array} \right) \end{array} \right] > 0 \end{array} \qquad (17)$$

The second order condition is not necessarily met for all values of T_r, and parameter values. In the large brackets, the first and second lines are positive, but the third can be negative. We can still use this condition to test whether a local extreme is a maximum or a minimum, and whether a local optimal is global.

3.3. An Effective Replacement Policy

Eq (9) suggests a monitoring methodology for replacement. It says the optimal cycle time is the time at which the marginal cost is equal to the average cost. For a given system, we can calculate and monitor the marginal cost as the system ages. In cases that we have the historical behavior of similar systems, we can use this information to calculate distributions on the expected cost at optimal replacement. This will be our best estimate of the behavior expected from a future replacement. Eq (9) suggests that we can monitor the marginal cost, and replace the system once this marginal cost crosses the average cost point. We can also use an adjusted average cost point to account for skewed behavior in realized average costs. A future topic of research is to study the performance of this policy, and various versions of it, compared to other known policies that apply to systems under the same constraints.

This approach is optimal under certain conditions. Later in this section, we will show that this policy is the first order condition for replacements with a new technology. Because the new technology can be identical to the old without loss of applicability, the result applies to this case also.

3.4. Replacing with a New Technology

The system or component of interest is now obsolete, so we will replace it with a new technology, possibly having a different cost and hazard function. Our question is in two parts:

1. When do we replace the current technology, and
2. What is the age replacement interval for the new technology.

As we will demonstrate, the answers to these questions relate under certain conditions.

The total expected cost per unit of time for one cycle of the obsolete technology, and m cycles of new technology is

$$C(T_r,\overline{T}_r) = \frac{m\left(\overline{C}_2 + \int_0^{\overline{T}_r} \overline{h}(u)\overline{C}_1(u)\delta u\right) + C_2 + \int_0^{T_r} h(u)C_1(u)\delta u}{m\overline{T}_r + T_r} \quad (18)$$

Note that, due to the problem formulation here, we can allow continuous values of T_r and $\overline{T}_r$ for the optimal solution. Therefore, we do not rely on Theorems 1 or 2.

The first order conditions are

$$\frac{\delta C}{\delta \overline{T}_r} = m\frac{\overline{h}(\overline{T}_r)\overline{C}_1(\overline{T}_r)}{m\overline{T}_r + T_r} - m\frac{C(T_r,\overline{T}_r)}{m\overline{T}_r + T_r} = 0 \quad (19)$$

$$\frac{\delta C}{\delta T_r} = \frac{h(T_r)C_1(T_r)}{m\overline{T}_r + T_r} - \frac{C(T_r,\overline{T}_r)}{m\overline{T}_r + T_r} = 0 \quad (20)$$

Notice that these conditions are of the same form. The second order conditions are

$$\frac{\delta^2 C}{\delta \overline{T}_r^2} = m\left[\frac{\overline{h}'(\overline{T}_r)\overline{C}_1(\overline{T}_r) + \overline{h}(\overline{T}_r)\overline{C}'_1(\overline{T}_r)}{m\overline{T}_r + T_r}\right] \quad (21)$$

$$\frac{\delta^2 C}{\delta T_r^2} = \left[\frac{h'(T_r)C_1(T_r) + h(T_r)C'_1(T_r)}{m\overline{T}_r + T_r}\right] \quad (22)$$

$$\frac{\delta^2 C}{\delta \overline{T}_r T_r} = -m\frac{\overline{h}(\overline{T}_r)\overline{C}_1(\overline{T}_r)}{m\overline{T}_r + T_r} + m\frac{C(T_r,\overline{T}_r)}{m\overline{T}_r + T_r} - m\frac{h(T_r)C_1(T_r)}{m\overline{T}_r + T_r} + m\frac{C(T_r,\overline{T}_r)}{m\overline{T}_r + T_r} \quad (23)$$

By the first order conditions, eq (23) is 0. When the hazard rate and cost rate are both positive, and at least one of them is increasing in time, then the Hessian is positive definite at the point meeting the first order condition. Thus, the first order conditions provide local minimums, if not global minimums.

Eqs (18) - (20) tell us several important facts:

- The replacement cycle times for the obsolete and new technologies depend upon one another. In fact, the relationship is

$$\overline{h}(\overline{T}_r)\overline{C}_1(\overline{T}_r) = C(T_r,\overline{T}_r) = h(T_r)C_1(T_r) \quad (24)$$

- As m increases, $C(T_r,\overline{T}_r)$ approaches $C(\overline{T}_r)$. As we plan for more cycles of the new technology, the obsolete technology has less influence on the cycle time of the new technology.
- For very small m, the cycle times have strong influence on one another.

Independent calculation of the cycle times for the two technologies will be sub-optimal. In each case, unless the new technology is mathematically identical to the obsolete, the optimal cycle time for the obsolete technology will be different in light of the new technology. The optimal solution is to calculate both cycle times concurrently with eqs (19) – (20).

In cases where we cannot obtain closed form solutions for T_r and $\overline{T}_r$, select $\overline{T}_r$ independently of the obsolete technology, and use this as an initial solution. Then use a method like Newton-Raphson to converge on solutions. Future work may be to prove the conditions for convergence. For large m and mathematically-similar technologies, the initial solution may be close to the optimal.

If we assume that the costs for both technologies are constant and equal, then from eq (24) we know that $\overline{h}(\overline{T}_r) = h(T_r)$. If we assume a Weibull hazard function, then the first order condition for the new technology becomes

$$m\overline{\lambda}^{\overline{\beta}}(\overline{\beta}-1)\overline{T}_r^{\overline{\beta}} + \left[\frac{\overline{\lambda}^{\overline{\beta}}\overline{\beta}}{\lambda^{\beta}\beta}\right]^{\frac{\beta}{\beta-1}}\overline{\lambda}^{\overline{\beta}}\overline{\beta}\left[\overline{T}_r^{\overline{\beta}-1}\right]^{\frac{2\beta-1}{\beta-1}} - \lambda\beta\left[\frac{\overline{\lambda}^{\overline{\beta}}\overline{\beta}}{\lambda^{\beta}\beta}\right]\left[\overline{T}_r^{\overline{\beta}-1}\right]^{\frac{\beta}{\beta-1}} - \frac{m\overline{C}_2 + C_2}{C_1} = 0 \quad (25)$$

which does not involve T_r. Therefore, we can calculate $\overline{T}_r$, then use the relationship that

$$T_r = \left(\frac{\overline{\lambda}^{\overline{\beta}}\overline{\beta}\overline{T}_r^{\overline{\beta}-1}}{\lambda^{\beta}\beta}\right)^{\frac{1}{\beta-1}} \quad (26)$$

Now lets extend the two-cycle time problem to one with a single cycle of the current obsolete technology, and possibly the new technology for an unknown number of cycles, with a total time of all cycles equal to T_h. The total expected cost over a long horizon is

$$TC(\overline{T}_r, T_r) = \frac{T_h - T_r}{\overline{T}_r}\left[\overline{C}_2 + \int_0^{\overline{T}_r} \overline{h}(u)\overline{C}_1(u)du\right] + C_2 + \int_0^{T_r} h(u)C_1(u)du \tag{27}$$

with the first ratio restricted to be an integer. But as before, we explore the problem with this ratio being continuous, and rely on search methods for finding the discrete optimal solutions.

The first order conditions are

$$\frac{\delta TC}{\delta \overline{T}_r} = \frac{T_h - T_r}{\overline{T}_r}\left[\overline{h}(\overline{T}_r)\overline{C}_1(\overline{T}_r) - \frac{\overline{C}_2 + \int_0^{\overline{T}_r} \overline{h}(u)\overline{C}_1(u)du}{\overline{T}_r}\right] = 0 \tag{28}$$

$$\frac{\delta TC}{\delta T_r} = h(T_r)C_1(T_r) - \frac{\overline{C}_2 + \int_0^{\overline{T}_r} \overline{h}(u)\overline{C}_1(u)du}{\overline{T}_r} = 0 \tag{29}$$

The second order conditions are too complex to demonstrate valuable insight. But if we include the first order conditions, they simplify to

$$\frac{\delta^2 TC}{\delta \overline{T}_r^2} = \frac{T_h - T_r}{\overline{T}_r}\left[\overline{h}'(\overline{T}_r)\overline{C}_1(\overline{T}_r) + \overline{h}(\overline{T}_r)\overline{C}'_1(\overline{T}_r)\right] \tag{30}$$

$$\frac{\delta^2 TC}{\delta T_r^2} = h'(T_r)C_1 + h(T_r)C'_1(T_r) \tag{31}$$

$$\frac{\delta^2 TC}{\delta \overline{T}_r T_r} = 0 \tag{32}$$

which, by our assumptions, gives a positive definite Hessian, and guarantees our solutions to the first order conditions are local minimum cost solutions, if not global.

The first order condition of the new technology, eq (28), is equivalent to the first order condition of eq (9). In fact, we can interpret eq (28) as we did eq (9). It says the average cost of the new technology equals the marginal cost of the old at the optimal time. Due to this interpretation of eq (28), we have proof that the policy described in section 3.3 can be optimal under the stated conditions. But in that application, we must select T_r differently than in this model.

If the second order conditions are met, we can set the cycle time of the new technology independent of the obsolete technology by applying eqn (28). Then, because of the form of the first order conditions, eqs (28) – (29), we apply the condition of eq (24) to find the cycle time for the current technology. However, this will not guarantee optimal solutions to fit our finite mission time interval T_h. If T_h is uncertain, this solution may be sufficient. If not, the nature of the constraint makes this problem difficult. We leave the solution to this problem for future work.

4. EVALUATING REPLACEMENT DECISIONS UNDER LIMITED REPLACEMENT RESOURCES

The models in this paper, and other modeling literature, assume availability of unlimited resources for replacements. Reality often is that corporations can ignore optimal replacement in favor of better investments, or be ignorant to optimal replacement policies altogether. Under this reality, to minimize costs, we must now find the systems or components that will yield the best return on investment from replacement.

Let the hazard rate follow a Weibull distribution. The expected present value of the total cost from a current age T_0 up to a future time T of either an existing component, or a new component, respectively is

$$EC_{old} = C_1 \int_{T_0}^{T+T_0} \frac{\lambda^\beta \beta t^{\beta-1}}{(1+i)^{t-T_0}} dt \tag{33}$$

$$EC_{new} = C_2 + C_1 \int_{T_0}^{T+T_0} \frac{\lambda^\beta \beta t^{\beta-1}}{(1+i)^{t-T_0}} dt \tag{34}$$

With equipment in-use, T_0 is the current age of the component or system. With a potential replacing equipment, $T_0 = 0$. If we further restrict β to be a positive integer, and assume that we replace the old equipment with new equipment of the same type, then the difference in cost over a finite time t is

$$EC_{old} - EC_{new} = -C_2 + \frac{C_1 \lambda^\beta \beta!}{(\ln(1+i))^\beta} \times \sum_{s=1}^{\beta-1}\left[\frac{(T_0 \ln(1+i))^s}{s!} + \frac{(t\ln(1+i))^s}{s!(1+i)^t} - \frac{((t+T_0)\ln(1+i))^s}{s!(1+i)^t}\right] \tag{35}$$

If we instead want to ignore the net present value, but allow cost to be a function of age, then the difference in cost over a finite time t becomes

$$EC_{old} - EC_{new} = -C_2 + \frac{C_1 \lambda^\beta \beta!}{(\ln(1+i))^\beta} \times \sum_{s=1}^{\beta-1}\left[\frac{(T_0 \ln(1+i))^s}{s!(1+i)^{T_0}} + \frac{(t\ln(1+i))^s}{s!(1+i)^t} - \frac{((t+T_0)\ln(1+i))^s}{s!(1+i)^{t+T_0}}\right] \tag{36}$$

In either case, if β is real, use an approximation method to solve the integrals.

Depending on the nature of the problem we investigate, we choose either eq (35) or eq (36) as an estimate of cost impact. As calculated, negative values are investment opportunities. The next step is to select the various investment opportunities under a limited budget. This is a standard knapsack problem.

The planning horizon can impact the results significantly. Therefore, select this carefully.

APPENDIX

Proof of Theorem 2:

If $f(x) = f(y)$ or $f(y) = f(z)$, then the proof is obvious. We can simply choose the solution of y if also $x = y$ or $y = z$, or choose either x or z if their values equal the minimum. Therefore, we have yet to prove the case of $x < y < z$.

Let's prove by contradiction. Assume the theorem is incorrect, that there is an integer w such that $w < x < y < z$, and $f(w) \le f(x)$. The proof for the case of $x < y < z < w$ is the same, with x and z interchanged and w being greater instead of less than. We can write $x = \alpha w + (1-\alpha)y$ for $0 < \alpha < 1$. Because $w < x < y$, we know that $\alpha \ne 0,1$. Because y is the continuous value minimum, we know $f(y) < f(x)$.

Therefore,

$\alpha f(w) + (1-\alpha) f(y) < f(x) = f(\alpha w + (1-\alpha)y)$. But $f(*)$ is convex, so this cannot be true. Therefore, the theorem must be true.

Proof of Theorem 3:

The cost function X provides a present value. If we change the reference time to $-t$ or t, we can take the net present value of this now future or past amount as $X(1+i)^{-t}$ or $X(1+i)^{t}$, respectively. Recall that X was defined before t so X is not a function of t. Also recall that the net present value at the new current time t is a function of the new current time and discount rate, and is a multiplication of the cost X. Therefore, the total cost at the new time is $TC(T_r) \times (1+i)^{-t}$ or $TC(T_r) \times (1+i)^{t}$. The added term $(1+i)^{-t}$ or $(1+i)^{t}$ is a positive constant in the optimal cycle time calculations. Therefore, the first and second order conditions are equivalent, and the optimal solution T_r is the same, regardless of the added term involving i and t. Therefore, the T_r will be the same for every cycle, regardless of the discount rate, or the time at which the cycle begins.

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Jason Rupe completed the *BS* and *MS* degrees from Iowa State University in Industrial Engineering. After transferring to Texas A&M University, he received the *PhD* in Industrial Engineering under an Operations Research program. In academia, he conducted research in command and control, communication and information network performance modeling, performability modeling of flexible systems, and tool changing problems. He now works at U S WEST Advanced Technologies as a Member of the Technical Staff in the Mathematical and Statistical Modeling Group. He acts as an internal consultant to operations problems and system designs, focusing on reliability issues. He applies the skills of reliability, quality, stochastic processes, statistics, and industrial engineering to U S WEST problems. He is a Member of IEEE and a Senior Member of IIE. He is an Associate Editor of IEEE *Transactions on Reliability*.